



Case Study 02

Basic data

Title of Research Teaching Unit	Applications of Exponential Functions
Unit created by	BORG Deutsch-Wagram (AT)
Based on Concept	Real-life exponential models
Concept created by	Škola da Vinci (CZ)
Date of teaching	20.11.2023
Number of students in class	11
Number of migrant students	1

1. Unit Description of Research Teaching Unit

Title	Applications of Exponential Functions
Age of students/grade	16 years
Length of the unit	1 x 50 minutes
Mathematical topic	Nonlinear functions
Prerequisite knowledge	<ul style="list-style-type: none">• Properties of Exponential Functions• Interpretation of graphs
Materials needed	<ul style="list-style-type: none">• Work sheet “Applications of Exponential Functions” (1 copy per student)• Index cards “Growth and Decay” (a few copies, if possible laminated)• Eventually: Calculator
Objectives	<ul style="list-style-type: none">• Applying exponential functions in real-world problems• Consolidating knowledge of exponential functions• Working with mathematical models of real-word problems
Notes on work with migrant students	<ul style="list-style-type: none">•••

Lesson plan:

Before the start of the lesson, the copies of the index cards are distributed at a few places in the classroom. Each place should have a full set of the index cards. These places serve as “help stations” where students can look up information that will help them solve the tasks of the work sheet.

Time	Method	Content and comments
ca. 10 min	Exposition and Teacher-student-communication	The teacher motivates the students for the topic of the lesson and describes the importance of percentages in everyday life. The teacher, together with the students, repeats the main concepts and properties of exponential functions.
ca. 25 min	Partner work	The students solve the tasks of the work sheet in partner work (2 students). The teacher supports the teams if required.
ca. 15 min	Teacher-student-communication	Comparison of the results

Applications of Exponential Functions – Work sheet

Locusts are insects that are widely spread in many countries of the world. There are many species of locusts, but they all have powerful hind legs allowing them to leap.



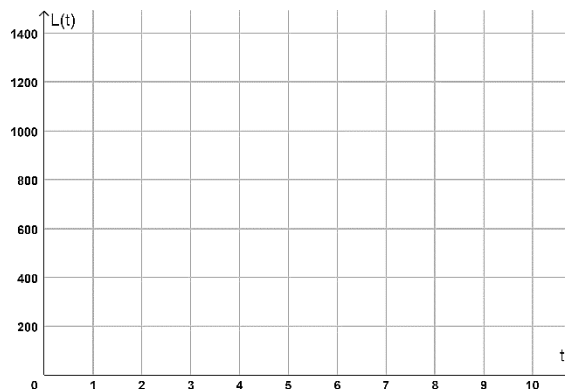
The larvae of locusts hatch at the beginning of spring when it gets warmer. They feed on the young shoots of plants and trees. The population of locusts increases exponentially when conditions are ideal for them.

Task 1:

At the beginning of spring, 100 locusts of species A are counted in a specified area.

It is assumed that the population will grow by 45% per day over the next few days. The time since the beginning of spring is given as t in days and the total number of locusts as $L(t)$.

- a) Create a graphical representation of the development of the locust population within one week. Write down the calculated values of each single day.



- b) Set up an equation for the function $L(t)$ in the form $L(t) = L_0 \cdot a^t$.
- c) Calculate when the locust population has increased eightfold and check your result with the graph from task a). Mark the calculated value in task a).

Task 2:

In the same specified area, 20 locusts of species B are counted two days after the start of spring. Eight days later the count is repeated and 47 locusts are counted. The time since the beginning of spring is given as t in days and the total number of locusts as $H(t)$.

- a) Set up an equation for $H(t)$ in the form $H(t) = H_0 \cdot e^{\lambda \cdot t}$.
- b) Determine by what percentage the locust population increases in one day.
- c) Determine by what percentage the locust population increases in one hour.

Applications of Exponential Functions – Work sheet with solutions

Locusts are insects that are widely spread in many countries of the world. There are many species of locusts, but they all have powerful hind legs allowing them to leap.



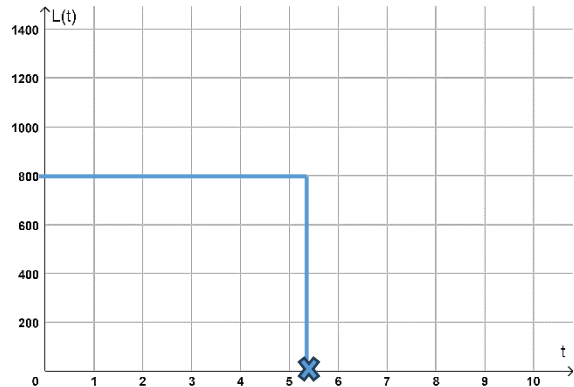
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Task 1:

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It is assumed that the population will grow by 45% per day over the next few days. The time since the beginning of spring is given as t in days and the total number of locusts as $L(t)$.

- a) Create a graphical representation of the development of the locust population within one week. Write down the calculated values of each single day.



- b) Set up an equation for the function $L(t)$ in the form $L(t) = L_0 \cdot a^t$. $L(t) = 100 \cdot 1.45^t$
- c) Calculate when the locust population has increased eightfold and check your result with the graph from task a). Mark the calculated value in task a).

$$\begin{aligned} 8 \cdot 100 &= 100 \cdot 1.45^t & |:100 \\ 8 &= 1.45^t & |\log \quad \cdot \\ \log(8) &= t \cdot \log(1.45) & |: \log(1.45) \\ t &= \frac{\log(8)}{\log(1.45)} \cong 5.60 \text{ days} \end{aligned}$$

Task 2:

In the same specified area, 20 locusts of species B are counted two days after the start of spring. Eight days later the count is repeated and 47 locusts are counted. The time since the beginning of spring is given as t in days and the total number of locusts as $H(t)$.

- a) Set up an equation for $H(t)$ in the form $H(t) = H_0 \cdot e^{\lambda \cdot t}$.
- b) Determine by what percentage the locust population increases in one day.
- c) Determine by what percentage the locust population increases in one hour.

Two days after the start of spring one gets: $H(2) = 20$. Eight days later one gets: $H(10) = 47$

$$H(10) = H(2) \cdot a^8 \rightarrow 47 = 20 \cdot a^8 \rightarrow a = \sqrt[8]{\frac{47}{20}} \cong 1.1127$$

$$H(2) = H(0) \cdot a^2 \rightarrow H(0) = \frac{H(2)}{a^2} = \frac{20}{1.1127^2} \cong 16 \text{ locusts}$$

$$\text{Equation: } H(t) = 16 \cdot 1.1127^t$$

$$a = e^\lambda \rightarrow 1.1127 = e^\lambda \rightarrow \lambda = \ln(1.1127) \cong 0.1068$$

$$\text{Equation: } H(t) = 16 \cdot e^{0.1068 \cdot t} \quad \dots \text{Solution of task a)}$$

Solution of task b): The locust population grows by 11.27% per day

Solution of task c): $1.1127^{\frac{1}{24}} \cong 1.0045$ The locust population grows by 0.45% per hour.

Index cards to cut and laminate:

Laws of Growth and Decay:

Exponential functions can have an arbitrary positive base. In the sciences, Euler's number e is frequently used. Aside from the general representation $f(x) = c \cdot a^x$, often $f(x) = c \cdot e^{\lambda x}$ with $\lambda \in \mathbb{R}$ is used.

	Exponential growth	Exponential decrease / decay
Arbitrary base a	$N(t) = N_0 \cdot a^t$ with $a > 1, t \geq 0$ a is called growth factor	$N(t) = N_0 \cdot a^t$ with $0 < a < 1, t \geq 0$ a is called factor of decrease
Base e	$N(t) = N_0 \cdot e^{+\lambda t}$ mit $\lambda > 0, t \geq 0$ λ is called growth constant	$N(t) = N_0 \cdot e^{-\lambda t}$ mit $\lambda > 0, t \geq 0$ λ is called decay constant

λ ... „lambda“, Greek letter

Examples:

	Exponential growth	Exponential decrease / decay
Arbitrary base a	$N(t) = 20 \cdot 1.23^t$	$N(t) = 20 \cdot 0.23^t$
Base e	$N(t) = 20 \cdot e^{0.4 t}$	$N(t) = 20 \cdot e^{-0.4 t}$

Important remarks:

- The **initial value** is denoted by N_0 , which means: $N_0 = N(t = 0)$
- How much a value changes by unit of time can only be read off the law of growth or decay, if it is in the form $f(x) = c \cdot a^x$ or $N(t) = N_0 \cdot a^t$.

Demonstration task 1:

Express the growth law $N(t) = 1500 \cdot 1.05^t$ using the base e !

The two forms of the growth law are:

$$N(t) = N_0 \cdot a^t \text{ and } N(t) = N_0 \cdot e^{+\lambda t}$$

Comparing those two, one can see that

$a = e^\lambda$. This can now be filled in with the numbers from the task.

$$a = e^\lambda \quad / \text{ Filling in } a = 1.05$$

$$1.05 = e^\lambda \quad / \ln$$

$$\ln(1.05) = \lambda \cdot \underbrace{\ln(e)}_{=1}$$

$$\ln(1.05) = \lambda \quad / \text{ Calculate } \ln(1.05)$$

$$0.049 \approx \lambda$$

The growth law (with base e) is now:

$$N(t) = 1500 \cdot e^{0.049 \cdot t}$$

Demonstration task 2:

A quantity grows according to the growth law $N(t) = 490 \cdot e^{0.04879 \cdot t}$, where t is measured in years. By what percentage does the quantity increase per year?

To find out by what percentage the quantity increases per year, one has to use the form $N(t) = N_0 \cdot a^t$. Only there one can read off the percentage.

The two forms of the growth law have to be compared. Those are:

$$N(t) = N_0 \cdot a^t \text{ and } N(t) = N_0 \cdot e^{+\lambda t}$$

Comparing those two, one can see that $a = e^\lambda$. This can now be filled in with the numbers from the task.

The growth law (with base a) is now:

$$a = e^\lambda \quad / \text{ Filling in } e^{0.04879}$$

$$a = e^{0.04879} \quad / \text{ Calculate } e^{0.04879}$$

$$a \approx 1.05$$

$$N(t) = 490 \cdot 1.05^t$$

From this equation one can read off that the **quantity increases by 5 % per year**.

Setting up a growth or decay law, using two known values

Demonstration task:

The timber stock of a forest was estimated at 150 000 cubic metres in 2005. Five years later, in 2010, the estimate was 168 000 cubic metres. Exponential growth is assumed. Set up a growth law for the timber stock h with $h(t) = h_0 \cdot a^t$ as a function of the number of years t that have passed since 2000.

Step 1: Calculating parameter a

From the tasks' text, we know that $t = 0$ means the year 2000 ($h_0 = h(0)$).

Also, we know two values:

The timber stock in 2005 is 150 000 m³ $\rightarrow h(5) = 150\,000$

The timber stock in 2010 is 168 000 m³ $\rightarrow h(10) = 168\,000$

The two values $h(5)$ and $h(10)$ and the property

$f(x+h) = f(x) \cdot a^h$ are used to calculate the parameter a .

Note: For solving the tasks of the work sheet, round the parameter a to 4 decimals, unless it is stated otherwise!

$$h(5) = 150\,000$$

$$h(10) = 168\,000$$

$$h(10) = h(5) \cdot a^5$$

$$168\,000 = 150\,000 \cdot a^5$$

$$a^5 = \frac{168\,000}{150\,000}$$

$$a = \sqrt[5]{\frac{168\,000}{150\,000}}$$

$$a \approx 1.0229$$

Step 2: Calculating the value of h_0

To calculate h_0 , we again use the property

$$f(x+h) = f(x) \cdot a^h.$$

$h(0)$ can be found by filling in the known values.

Here we round to integers, since the tasks' text also uses integers.

$h(0) \approx 133945$ gives the timber stock in the year 2000.

$$h(5) = h(0) \cdot a^5$$

$$150\,000 = h(0) \cdot 1.0229^5$$

$$\frac{150\,000}{1.0229^5} = h(0)$$

$$h(0) \approx 133945$$

Step 3: Setting up the growth law

The task asks us to use the form $h(t) = h_0 \cdot a^t$. Filling in the values for h_0 and a that we just calculated, the growth law can be set up:

$$h(t) = h_0 \cdot a^t$$

Growth law:

$$h(t) = 133\,945 \cdot 1.0229^t$$

Calculating values of t in the exponent

Demonstration task:

A bacterial culture grows by 80 % every hour. At the beginning, it consists of 800 bacteria. Calculate **when** the number of bacteria has increased **tenfold**!

Step 1: Setting up the growth law

Since the bacterial culture grows by 80 %, one has to multiply with the factor 1.80 for every hour. Hence the parameter a is

$$a = 1.80 \text{ (Explanation: } 100\% + 80\% = 180\% = \frac{180}{100} = 1.80)$$

Since at the beginning there are 800 bacteria, one gets: $N_0 = 800$

$$N(t) = N_0 \cdot a^t$$

Growth law:

$$N(t) = 800 \cdot 1.80^t$$

Step 2: Calculating the time t at which the number of bacteria has increased tenfold $\rightarrow$ we are looking for the exponent $t \rightarrow$ we have to use the logarithm

At the beginning, there are 800 bacteria. If those increase tenfold, there are $10 \cdot 800$ bacteria $\rightarrow N(t) = 10 \cdot 800$

To find out the time t , $10 \cdot 800$ has to be filled into the growth law, and then t can be calculated.

$$N(t) = 8000 \quad \text{und}$$

$$N(t) = 800 \cdot 1.80^t$$

$$10 \cdot 800 = 800 \cdot 1.80^t \quad | :800$$

$$10 = 1.80^t \quad | \log$$

$$\log(10) = \log(1.80^t)$$

$$\log(10) = t \cdot \log(1.80) \quad | : \log(1.80)$$

$$t = \frac{\log(10)}{\log(1.80)}$$

$$t \approx 3.92 \text{ h}$$

One can see that the time t is independent of the initial value of the number of bacteria. It only depends on the growth factor a !

2. Interview Analysis

The teacher teaching this unit has more than 30 years teaching experience. Her second subject is Geography. She did not receive any particular training about teaching in multicultural classrooms.

The teacher particularly commented on the motivational effect that this teaching unit had on the students. She described the students in this class as “not really motivated”, and this was also clearly visible in the first few minutes of the lesson, when student behaviour was discussed (apparently not for the first time). While introducing the topic and presenting the demonstration task, the motivation did not seem to improve much. However, once the students received the work sheets and started working on the actual tasks, the motivation increased considerably, and the students worked enthusiastically. The teacher commented on her being surprised about that, and assumed that the context of the unit and the reality of the tasks was responsible for that.

The teacher mentioned that this unit was a good repetition of exponential functions, to better prepare the students for the final examination (at the end of this school year). She commented that students would probably not have engaged with the topic so much if she would have just repeated the equations, or used the “standard context” of radioactive decay.

3. Improvement Suggestions

The teacher did not recommend any specific changes. She just commented that perhaps subtasks a) and b) could possibly be exchanged with each other, since the calculation of points of the graph is probably easier if the function equation has been established before.